A stochastic calculus proof of the CLT for the L^2 modulus of continuity of local time

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Abstract

We give a stochastic calculus proof of the Central Limit Theorem

$$\frac{\int (L_t^{x+h} - L_t^x)^2 \, dx - 4ht}{h^{3/2}} \stackrel{\mathcal{L}}{\Longrightarrow} c \left(\int (L_t^x)^2 \, dx \right)^{1/2} \eta$$

as $h \to 0$ for Brownian local time L_t^x. Here η is an independent normal random variable with mean zero and variance one.

1   Introduction

In [4] we obtain almost sure limits for the L^p moduli of continuity of local times of a very wide class of symmetric Lévy processes. More specifically, if $\{L_t^x \, ; \, (x,t) \in R^1 \times R_+^1\}$ denotes Brownian local time then for all $p \geq 1$, and all $t \in R_+$

$$(1.1) \qquad \lim_{h \downarrow 0} \int_a^b \left| \frac{L_t^{x+h} - L_t^x}{\sqrt{h}} \right|^p \, dx = 2^p E(|\eta|^p) \int_a^b |L_t^x|^{p/2} \, dx$$

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for all a, b in the extended real line almost surely, and also in L^m, $m \geq 1$. (Here η is normal random variable with mean zero and variance one.) In particular when $p = 2$ we have

$$(1.2) \qquad \lim_{h \downarrow 0} \int \frac{(L_t^{x+h} - L_t^x)^2}{h} \, dx = 4t, \qquad \text{almost surely.}$$

We refer to $\int (L_t^{x+h} - L_t^x)^2 \, dx$ as the L^2 modulus of continuity of Brownian local time.

In our recent paper [1] we obtain the central limit theorem corresponding to (1.2).

Theorem 1 *For each fixed t*

$$(1.3) \qquad \frac{\int (L_t^{x+h} - L_t^x)^2 \, dx - 4ht}{h^{3/2}} \overset{\mathcal{L}}{\Longrightarrow} c \left(\int (L_t^x)^2 \, dx \right)^{1/2} \eta$$

as $h \to 0$, with $c = \left(\frac{64}{3} \right)^{1/2}$. Equivalently

$$(1.4) \qquad \frac{\int (L_t^{x+1} - L_t^x)^2 \, dx - 4t}{t^{3/4}} \overset{\mathcal{L}}{\Longrightarrow} c \left(\int (L_1^x)^2 \, dx \right)^{1/2} \eta$$

as $t \to \infty$. Here η is an independent normal random variable with mean zero and variance one.

It can be shown that

$$(1.5) \qquad E \left(\int (L_t^{x+1} - L_t^x)^2 \, dx \right) = 4 \left(t - \frac{2t^{1/2}}{\sqrt{2\pi}} \right) + O(1).$$

so that (1.4) can be written as

$$(1.6) \frac{\int (L_t^{x+1} - L_t^x)^2 \, dx - E \left(\int (L_t^{x+1} - L_t^x)^2 \, dx \right)}{t^{3/4}} \overset{\mathcal{L}}{\Longrightarrow} c \left(\int (L_1^x)^2 \, dx \right)^{1/2} \eta$$

with a similar statement for (1.3).

Our proof of Theorem 1 in [1] is rather long and involved. We use the method of moments, but rather than study the asymptotics of the moments of (1.3), which seem intractable, we study the moments of the analogous expression where the fixed time t is replaced by an independent exponential

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time of mean $1/\lambda$. An important part of the proof is then to 'invert the Laplace transform' to obtain the asymptotics of the moments for fixed t.

The purpose of this paper is to give a new and shorter proof of Theorem 1 using stochastic integrals, following the approach of [9, 10]. Our proof makes use of certain differentiability properties of the double and triple intersection local time, $\alpha_{2,t}(x)$ and $\alpha_{3,t}(x,y)$, which are formally given by

$$(1.7) \qquad \alpha_{2,t}(x) = \int_0^t \int_0^s \delta(W_s - W_r - x)\, dr\, ds$$

and

$$(1.8) \quad \alpha_{3,t}(x,y) = \int_0^t \int_0^s \int_0^r \delta(W_r - W_{r'} - x)\delta(W_s - W_r - y)\, dr'\, dr\, ds.$$

More precisely, let $f(x)$ be a smooth positive symmetric function with compact support and $\int f(x)\, dx = 1$. Set $f_\epsilon(x) = \frac{1}{\epsilon}f(x/\epsilon)$. Then

$$(1.9) \qquad \alpha_{2,t}(x) = \lim_{\epsilon \to 0} \int_0^t \int_0^s f_\epsilon(W_s - W_r - x)\, dr\, ds$$

and

$$
\begin{aligned}
& \alpha_{3,t}(x,y) \\
(1.10) \quad & = \lim_{\epsilon \to 0} \int_0^t \int_0^s \int_0^r f_\epsilon(W_r - W_{r'} - x)f_\epsilon(W_s - W_r - y)\, dr'\, dr\, ds
\end{aligned}
$$

exist almost surely and in all L^p, are independent of the particular choice of f, and are continuous in (x,y,t) almost surely, [6]. It is easy to show, see [8, Theorem 2], that for any measurable $\phi(x)$

$$(1.11) \qquad \int_0^t \int_0^s \phi(W_s - W_r)\, dr\, ds = \int \phi(x)\alpha_{2,t}(x)\, dx$$

and for any measurable $\phi(x,y)$

$$
\begin{aligned}
& \int_0^t \int_0^s \int_0^r \phi(W_r - W_{r'}, W_s - W_r)\, dr'\, dr\, ds \\
(1.12) \quad & = \int \phi(x,y)\alpha_{3,t}(x,y)\, dx\, dy.
\end{aligned}
$$

To express the differentiability properties of $\alpha_{2,t}(x)$ and $\alpha_{3,t}(x,y)$ which we need, let us set

$$(1.13) \qquad v(x) = \int_0^\infty e^{-s/2}p_s(x)\, ds = e^{-|x|}.$$

The following result is [8, Thorem 1].

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Theorem 2

(1.14) $$\gamma_{2,t}(x) =: \alpha_{2,t}(x) - tv(x)$$

and

(1.15) $$\gamma_{3,t}(x,y) =: \alpha_{3,t}(x,y) - \gamma_{2,t}(x)v(y) - \gamma_{2,t}(y)v(x) - tv(x)v(y)$$

are C^1 in (x,y) and $\nabla\gamma_{2,t}(x), \nabla\gamma_{3,t}(x,y)$ are continuous in (x,y,t).

Our new proof of Theorem 1 is given in Section 2.

Our original motivation for studying the asymptotics of $\int (L_t^{x+h} - L_t^x)^2\,dx$ comes from our interest in the Hamiltonian

(1.16) $$H_n = \sum_{i,j=1,\,i\neq j}^{n} 1_{\{S_i=S_j\}} - \frac{1}{2}\sum_{i,j=1,\,i\neq j}^{n} 1_{\{|S_i-S_j|=1\}},$$

for the critical attractive random polymer in dimension one, [2], where $\{S_n\,;\,n = 0,1,2,\dots\}$ is a simple random walk on Z^1. Note that $H_n = \sum_{x\in Z^1}\left(l_n^x - l_n^{x+1}\right)^2$, where $l_n^x = \sum_{i=1}^{n} 1_{\{S_i=x\}}$ is the local time for the random walk S_n.

2   A stochastic calculus approach

By [3, Lemma 2.4.1] we have that

(2.1) $$L_t^x = \lim_{\epsilon\to 0}\int_0^t f_\epsilon(W_s - x)\,ds$$

almost surely, with convergence locally uniform in x. Hence

$$\int L_t^{x+h} L_t^x\,dx$$

(2.2)
$$= \int \lim_{\epsilon\to 0}\left(\int_0^t f_\epsilon(W_s - (x+h))\,ds\right)\left(\int_0^t f_\epsilon(W_r - x)\,dr\right)dx$$

$$= \lim_{\epsilon\to 0}\int \left(\int_0^t f_\epsilon(W_s - (x+h))\,ds\right)\left(\int_0^t f_\epsilon(W_r - x)\,dr\right)dx$$

$$= \lim_{\epsilon\to 0}\int_0^t \int_0^t f_\epsilon * f_\epsilon(W_s - W_r - h)\,dr\,ds$$

$$= \lim_{\epsilon\to 0}\int_0^t \int_0^s f_\epsilon * f_\epsilon(W_s - W_r - h)\,dr\,ds$$

$$\qquad\qquad + \lim_{\epsilon\to 0}\int_0^t \int_0^r f_\epsilon * f_\epsilon(W_r - W_s + h)\,ds\,dr$$

$$= \alpha_{2,t}(h) + \alpha_{2,t}(-h).$$

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Note that

(2.3)
$$\int (L_t^{x+h} - L_t^x)^2 \, dx = 2 \left(\int (L_t^x)^2 \, dx - \int L_t^{x+h} L_t^x \, dx \right)$$

and thus

(2.4)
$$\int (L_t^{x+h} - L_t^x)^2 \, dx = 2 \left(2\alpha_{2,t}(0) - \alpha_{2,t}(h) - \alpha_{2,t}(-h) \right).$$

Hence we can prove Theorem 1 by showing that for each fixed t

(2.5)
$$\frac{2 \left(2\alpha_{2,t}(0) - \alpha_{2,t}(h) - \alpha_{2,t}(-h) \right) - 4ht}{h^{3/2}} \overset{\mathcal{L}}{\Longrightarrow} c\sqrt{\alpha_{2,t}(0)} \, \eta$$

as $h \to 0$, with $c = \left(\frac{128}{3} \right)^{1/2}$. Here we used the fact, which follows from (2.2), that $\int (L_1^x)^2 \, dx = 2 \, \alpha_{2,t}(0)$.

In proving (2.5) we will need the following Lemma. Compare Tanaka's formula, [5, Chapter VI, Theorem 1.2].

Lemma 1 *For any $a \in R^1$*

(2.6)
$$\alpha_{2,t}(a) = 2 \int_0^t (W_t - W_s - a)^+ \, ds - 2 \int_0^t (W_0 - W_s - a)^+ \, ds$$
$$-2(-a)^+ t - 2 \int_0^t \int_0^s 1_{\{W_s - W_r > a\}} \, dr \, dW_s.$$

Proof of Lemma 1: Set

(2.7)
$$g_\epsilon(x) = \int_0^\infty y f_\epsilon(x - y) \, dy$$

so that

(2.8)
$$g_\epsilon'(x) = \int_0^\infty y f_\epsilon'(x - y) \, dy = \int_0^\infty f_\epsilon(x - y) \, dy$$

and consequently

(2.9)
$$g_\epsilon''(x) = f_\epsilon(x).$$

Let

(2.10)
$$F_a(t, x) = \int_0^t g_\epsilon(x - W_s - a) \, ds.$$

Then by Ito's formula applied to $F_a(t, W_t)$ we have

(2.11)
$$\int_0^t g_\epsilon(W_t - W_s - a) \, ds - \int_0^t g_\epsilon(W_0 - W_s - a) \, ds$$
$$= \int_0^t g_\epsilon(-a) \, ds + \int_0^t \int_0^s g_\epsilon'(W_s - W_r - a) \, dr \, dW_s$$
$$+ \frac{1}{2} \int_0^t \int_0^s g_\epsilon''(W_s - W_r - a) \, dr \, ds.$$

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It is easy to check that locally uniformly

(2.12) $$\lim_{\epsilon \to 0} g_\epsilon(x) = x^+$$

and hence using (2.9) we obtain

(2.13)
$$\alpha_{2,t}(a) = 2 \int_0^t (W_t - W_s - a)^+ \, ds - 2 \int_0^t (W_0 - W_s - a)^+ \, ds$$
$$-2(-a)^+ t - 2 \lim_{\epsilon \to 0} \int_0^t \int_0^s g_\epsilon'(W_s - W_r - a) \, dr \, dW_s.$$

From (2.8) we can see that $\sup_x |g_\epsilon'(x)| \leq 1$ and

(2.14) $$\lim_{\epsilon \to 0} g_\epsilon'(x) = 1_{\{x>0\}} + \frac{1}{2} 1_{\{x=0\}}.$$

Thus by the dominated convergence theorem

(2.15) $$\lim_{\epsilon \to 0} \int_0^t E \left(\left(\int_0^s \left\{ g_\epsilon'(W_s - W_r - a) - 1_{\{W_s - W_r > a\}} \right\} dr \right)^2 \right) ds = 0$$

which completes the proof of our Lemma. □

If we now set

(2.16)
$$J_h(x) = 2x^+ - (x - h)^+ - (x + h)^+$$
$$= \begin{cases} -x - h & \text{if } -h \leq x \leq 0 \\ x - h & \text{if } 0 \leq x \leq h. \end{cases}$$

and

(2.17)
$$K_h(x) = 21_{\{x>0\}} - 1_{\{x>h\}} - 1_{\{x>-h\}}$$
$$= 1_{\{0<x\leq h\}} - 1_{\{-h<x\leq 0\}}$$

we see from Lemma 1 that

(2.18)
$$2\left\{2\alpha_t(0) - \alpha_t(h) - \alpha_t(-h)\right\} - 4ht$$
$$= 4 \int_0^t J_h(W_t - W_s) \, ds - 4 \int_0^t J_h(W_0 - W_s) \, ds$$
$$-4 \int_0^t \int_0^s K_h(W_s - W_r) \, dr \, dW_s.$$

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By (2.16)

$$(2.19) \qquad \int_0^t J_h(W_t - W_s)\, ds = \int J_h(W_t - x) L_t^x\, dx = O(h^2 \sup_x L_t^x)$$

and similarly for $\int_0^t J_h(W_0 - W_s)\, ds$. Hence to prove (2.5) it suffices to show that for each fixed t

$$(2.20) \qquad \frac{\int_0^t \int_0^s K_h(W_s - W_r)\, dr\, dW_s}{h^{3/2}} \overset{\mathcal{L}}{\Longrightarrow} \left(\frac{8}{3}\right)^{1/2} \sqrt{\alpha_{2,t}(0)}\, \eta$$

as $h \to 0$. Let

$$(2.21) \qquad M_t^h = h^{-3/2} \int_0^t \int_0^s K_h(W_s - W_r)\, dr\, dW_s.$$

It follows from the proof of Theorem 2.6 in [5, Chapter XIII], (the Theorem of Papanicolaou, Stroock, and Varadhan) that to establish (2.20) it suffices to show that

$$(2.22) \qquad \lim_{h \to 0} \langle M^h, W \rangle_t = 0$$

and

$$(2.23) \qquad \lim_{h \to 0} \langle M^h, M^h \rangle_t = \frac{8}{3} \alpha_{2,t}(0)$$

uniformly in t on compact intervals.

By (1.11), and using the fact that $K_h(x) = K_1(x/h)$, we have that

$$
\begin{aligned}
(2.24) \qquad \langle M^h, W \rangle_t &= h^{-3/2} \int_0^t \int_0^s K_h(W_s - W_r)\, dr\, ds \\
&= h^{-3/2} \int K_h(x) \alpha_{2,t}(x)\, dx \\
&= h^{-1/2} \int K_1(x) \alpha_{2,t}(hx)\, dx \\
&= \int_0^1 \frac{\alpha_{2,t}(hx) - \alpha_{2,t}(-hx)}{h^{1/2}}\, dx.
\end{aligned}
$$

But $v(hx) - v(-hx)$, so by Lemma 2 we have that

$$(2.25) \qquad \alpha_{2,t}(hx) - \alpha_{2,t}(-hx) = \gamma_{2,t}(hx) - \gamma_{2,t}(-hx) = O(h)$$

which completes the proof of (2.22).

We next analyze

$$(2.26) \qquad \langle M^h, M^h \rangle_t = h^{-3} \int_0^t \left(\int_0^s K_h(W_s - W_r) \, dr \right)^2 ds$$

$$= h^{-3} \int_0^t \left(\int_0^s K_h(W_s - W_r) \, dr \right) \left(\int_0^s K_h(W_s - W_{r'}) \, dr' \right) ds$$

$$= h^{-3} \int_0^t \left(\int_0^s \int_0^r K_h(W_s - W_{r'}) K_h(W_s - W_r) \, dr' \, dr \right) ds$$

$$+ h^{-3} \int_0^t \left(\int_0^s \int_0^{r'} K_h(W_s - W_r) K_h(W_s - W_{r'}) \, dr \, dr' \right) ds.$$

By (1.12) we have that

$$(2.27) \qquad \int_0^t \int_0^s \int_0^r K_h(W_s - W_{r'}) K_h(W_s - W_r) \, dr' \, dr \, ds$$

$$= \int_0^t \int_0^s \int_0^r K_h(W_s - W_r + W_r - W_{r'}) K_h(W_s - W_r) \, dr' \, dr \, ds$$

$$= \int \int K_h(x+y) K_h(y) \alpha_{3,t}(x,y) \, dx \, dy.$$

Using $K_h(x) = K_1(x/h)$ we have

$$(2.28) \qquad h^{-3} \int_0^t \int_0^s \int_0^r K_h(W_s - W_{r'}) K_h(W_s - W_r) \, dr' \, dr \, ds$$

$$= h^{-3} \int \int K_h(x+y) K_h(y) \alpha_{3,t}(x,y) \, dx \, dy$$

$$= h^{-1} \int \int K_1(x+y) K_1(y) \alpha_{3,t}(hx, hy) \, dx \, dy$$

$$= h^{-1} \int \int K_1(x) K_1(y) \alpha_{3,t}(h(x-y), hy) \, dx \, dy$$

$$= h^{-1} \int_0^1 \int_0^1 A_{3,t}(h, x, y) \, dx \, dy$$

where

$$(2.29) \qquad A_{3,t}(h, x, y) = \alpha_{3,t}(h(x-y), hy) - \alpha_{3,t}(h(-x-y), hy)$$
$$- \alpha_{3,t}(h(x+y), -hy) + \alpha_{3,t}(-h(x-y), -hy).$$

It remains to consider

$$(2.30) \qquad \lim_{h \to 0} \frac{A_{3,t}(h, x, y)}{h}.$$

We now use Lemma 2. Using the fact that $\gamma_{3,t}(x, y), \gamma_{2,t}(x)$ are continuously differentiable

$$
\begin{align}
(2.31) \quad & \gamma_{3,t}(h(x-y), hy) - \gamma_{3,t}(h(-x-y), hy) \\
&= h(x-y)\frac{\partial}{\partial x}\gamma_{3,t}(0, hy) - h(-x-y)\frac{\partial}{\partial x}\gamma_{3,t}(0, hy) + o(h) \\
&= 2hx\frac{\partial}{\partial x}\gamma_{3,t}(0, 0) + o(h)
\end{align}
$$

and similarly

$$
\begin{align}
(2.32) \quad & \gamma_{3,t}(-h(x-y), -hy) - \gamma_{3,t}(h(x+y), -hy) \\
&= -h(x-y)\frac{\partial}{\partial x}\gamma_{3,t}(0, -hy) - h(x+y)\frac{\partial}{\partial x}\gamma_{3,t}(0, -hy) + o(h) \\
&= -2hx\frac{\partial}{\partial x}\gamma_{3,t}(0, 0) + o(h)
\end{align}
$$

and these two terms cancel up to $o(h)$.

Next,

$$
\begin{align}
(2.33) \quad & \gamma_{2,t}(h(x-y))v(hy) - \gamma_{2,t}(h(-x-y))v(hy) \\
& \qquad\qquad\qquad + \gamma_{2,t}(-h(x-y))v(-hy) - \gamma_{2,t}(h(x+y))v(-hy) \\
&= h(x-y)\gamma'_{2,t}(0)v(0) - h(-x-y)\gamma'_{2,t}(0)v(0) \\
& \qquad\qquad - h(x-y)\gamma'_{2,t}(0)v(0) - h(x+y)\gamma'_{2,t}(0)v(0) + o(h) \\
&= o(h).
\end{align}
$$

On the other hand, using $v(x) = e^{-|x|} = 1 - |x| + O(x^2)$ we have

$$
\begin{align}
(2.34) \quad & v(h(x-y))\gamma_{2,t}(hy) - v(h(-x-y))\gamma_{2,t}(hy) \\
& \qquad\qquad\qquad + v(-h(x-y))\gamma_{2,t}(-hy) - v(h(x+y))\gamma_{2,t}(-hy) \\
&= -|h(x-y)|\gamma_{2,t}(0) + |h(-x-y)|\gamma_{2,t}(0) \\
& \qquad\qquad - |h(x-y)|\gamma_{2,t}(0) + |h(x+y)|\gamma_{2,t}(0) + o(h) \\
&= 2h(|x+y| - |x-y|)\gamma_{2,t}(0) + o(h).
\end{align}
$$

and similarly

$$
\begin{align}
(2.35) \quad & v(h(x-y))v(hy) - v(h(-x-y))v(hy) \\
& \qquad\qquad\qquad + v(-h(x-y))v(-hy) - v(h(x+y))v(-hy) \\
&= -|h(x-y)|v(0) + |h(-x-y)|v(0) \\
& \qquad\qquad - |h(x-y)|v(0) + |h(x+y)|v(0) + O(h^2) \\
&= 2h(|x+y| - |x-y|)v(0) + O(h^2).
\end{align}
$$

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Putting this all together and using the fact that $\alpha_{2,t}(0) = \gamma_{2,t}(0) + tv(0)$ we see that

$$(2.36) \qquad \int_0^1 \int_0^1 A_{3,t}(h, x, y)\, dx\, dy$$
$$= 2h\alpha_{2,t}(0) \int_0^1 \int_0^1 (|x + y| - |x - y|))\, dx\, dy + o(h).$$

Of course

$$(2.37) \qquad \int_0^1 \int_0^1 (|x + y| - |x - y|))\, dx\, dy$$
$$= \int_0^1 \int_0^x 2y\, dy\, dx + \int_0^1 \int_0^y 2x\, dx\, dy = \frac{2}{3}$$

so that

$$(2.38) \qquad \lim_{h \to 0} \frac{\int_0^1 \int_0^1 A_{3,t}(h, x, y)\, dx\, dy}{h} = \frac{4}{3}\alpha_{2,t}(0).$$

By (2.26) this gives (2.23). □

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